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MULTICOMMODITY FLOWS IN GRAPHS

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Suppose that G is a graph, and (s_i, t_i) $(1 \le i \le k)$ are pairs of vertices; and that each edge has a integer-valued capacity (≥ 0), and that $q_i \ge 0$ $(1 \le i \le k)$ are integer-valued demands. When is there a flow for each i, between s_i and t_i and of value q_i, such that the total flow through each edge does not exceed its capacity? Ford and Fulkerson solved this when $k=1$, and Hu when $k=2$. We solve it for general values of k, when G is planar and can be drawn so that $s_1, \dots, s_l, t_1, \dots, t_l$ are all on the boundary of a face and $s_{l+1}, \dots, s_k, t_{l+1}, \dots, t_k$ are all on the boundary of the infinite face or when $t_1 = \dots = t_l$ and G is planar and can be drawn so that $s_{l+1}, \dots, s_k, t_1, \dots, t_k$ are all on the boundary of the infinite face. This extends a theorem of Okamura and Seymour.

1. Introduction

Let $G=(V,E)$ be a graph (which means, in this paper, a finite undirected graph, possibly with multiple edges but without loops). Let $(s_1, t_1), \dots, (s_k, t_k)$ be pairs (not necessarily distinct) of vertices of G. Suppose that each edge $e \in E$ has a integer-valued capacity $w(e) \ge 0$ and that $q_i \ge 0$ $(1 \le i \le k)$ are integer-valued demands. When is the following statement true?

1.1. For $1 \le i \le k$ there is a flow F_i from s_i to t_i of value q_i, such that for each edge $e \in E$,

$$\sum_i |F_i(e)| \le w(e).$$

[Here $|F_i(e)|$ denotes the numerical value of the flow through e. Strictly speaking, $F_i(e)$ itself is not properly defined, because we have not oriented the edges of G.] This is called the multicommodity flow problem.

For sets $X_1, X_2 \subseteq V$, we let $\partial(X_1; X_2) \subseteq E$ be the set of edges with one end in X_1 and the other in X_2 and we denote $\partial(X_1; V-X_1)$ by $\partial(X_1)$. We let $D(X_1; X_2) \subseteq \{1, \dots, k\}$ be

$$\{i;\ 1 \le i \le k, \{s_i, t_i\} \cap X_j \ne \emptyset \text{ for } j=1,2\}$$

and we denote $D(X_1; V-X_1)$ by $D(X_1)$. It is clear that if 1.1 is true, then the following connectivity condition holds.

1.2. For each $X \subseteq V$,

$$\sum_{e \in \partial(X)} w(e) \geq \sum_{i \in D(X)} q_i.$$

Ford and Fulkerson [1] proved that when $k=1$, 1.2 is in fact equivalent to 1.1; and Hu [2] proved the same thing when $k=2$. They are not equivalent in general; a counterexample with $k=4$ is given in Fig. 1, where $w \equiv 1$ and $q_i = 1$ $(i=1,2,3,4)$. (Hu [2] gives a counterexample with $k=3$.) Okamura and Seymour [3] proved that for general values of k, 1.2 is equivalent to 1.1, when G is planar and can be drawn so that $s_1, \dots, s_k, t_1, \dots, t_k$ are all on the boundary of the infinite face.

Our main theorems are the following.

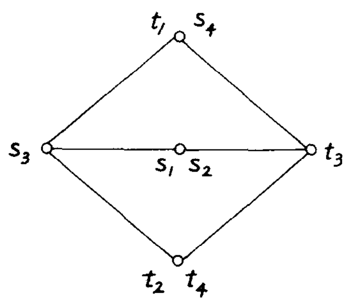

Fig. 1.

1.3. Theorem. *If G is planar and can be drawn in the plane so that $s_1, \dots, s_l, t_1, \dots, t_l$ are all on the boundary of a face and $s_{l+1}, \dots, s_k, t_{l+1}, \dots, t_k$ are all on the boundary of the infinite face, then* 1.1 *is equivalent to* 1.2.

1.4. Theorem. *If $t_1 = \cdots = t_l$ and G is planar and can be drawn in the plane so that $s_{l+1}, \dots, s_k t_1, \dots, t_k$ are all on the boundary of the infinite face, then* 1.1 *is equivalent to* 1.2.

1.5. Theorem. *If the hypothesis of* 1.3 *or* 1.4 *holds, and if* 1.2 *holds and w and q are even-integer-valued, then the flows F_i in* 1.1 *may be chosen integer-valued.*

By the counterexample in Fig. 1, we see that in Theorem 1.3 if one of $s_1, \dots, s_k, t_1, \dots, t_k$ does not satisfy the hypothesis in 1.4 if $l \geq 2$ and $t_1 \neq t_2$, then 1.1 is not equivalent to 1.2. We will in fact prove the conclusion of Theorem 1.5 under the weaker hypothesis that for each $X \subseteq V$

$$\sum_{e \in \partial(X)} w(e) - \sum_{i \in D(X)} q_i \quad (\geq 0)$$

is even. And to prove this, it suffices to deal with the case when $w \equiv 1$ and $q_i = 1$ $\forall i$ (by suitable deletion of edges and addition of parallel edges to G, and by suitable removal and repetition of pairs (s_i, t_i)). Thus to prove Theorems 1.3, 1.4, 1.5 and their strengthenings, we must prove the following.

1.6. Theorem. *Suppose that $G=(V,E)$ is a planar graph, and can be drawn in the plane so that the vertices $s_1, \dots, s_l, t_1, \dots, t_l$ are all on the boundary of a face and the vertices $s_{l+1}, \dots, s_k, t_{l+1}, \dots, t_k$ are all on the boundary of the infinite face; and suppose further that for each $X \subseteq V$,*

$$|\partial(X)| - |D(X)|$$

is even and non-negative. Then there exist edge-disjoint paths $P_1, \dots, P_k$ of G, such that P_i has ends s_i, t_i $(1 \le i \le k)$.

1.7. Theorem. *Suppose that $G=(V,E)$ is a planar graph, and can be drawn in the plane so that the vertices $s_{l+1}, \dots, s_k, t_1, \dots, t_k$ are all on the boundary of the infinite face, and $t_1 = \cdots = t_l$; and suppose further that for each $X \subseteq V$,*

$$|\partial(X)| - |D(X)|$$

is even and non-negative. Then there exist edge-disjoint paths $P_1, \dots, P_k$ of G, such that P_i has ends s_i, t_i $(1 \le i \le k)$.

2. Preliminaries

We say that $X \subseteq V$ is critical if $|\partial(X)| = |D(X)|$ and that X is elemental if $\langle X \rangle$ (which is the induced subgraph) and $\langle V - X \rangle$ are both connected. We prepare some lemmas.

2.1. Lemma. *Suppose that for each $X \subseteq V$, $|\partial(X)| \ge |D(X)|$. If $X_1, X_2 \subseteq V$ are both critical and $D(X_1 - X_2; X_2 - X_1) = \emptyset$, then $X_1 \cap X_2$, $X_1 \cup X_2$ are both critical and $\partial(X_1 - X_2; X_2 - X_1) = \emptyset$.*

Proof. By simple counting we have

$$|\partial(X_1 \cap X_2)| + |\partial(X_1 \cup X_2)| = |\partial(X_1)| + |\partial(X_2)| - 2|\partial(X_1 - X_2; X_2 - X_1)|,$$

and a similar equality holds with D instead of ∂. Thus

$$\begin{aligned} |D(X_1 \cap X_2)| + |D(X_1 \cup X_2)| &= |D(X_1)| + |D(X_2)| = |\partial(X_1)| + |\partial(X_2)| \\ &= |\partial(X_1 \cap X_2)| + |\partial(X_1 \cup X_2)| \\ &\quad + 2|\partial(X_1 - X_2; X_2 - X_1)|. \end{aligned}$$

Hence $\partial(X_1 - X_2; X_2 - X_1) = \emptyset$, $|\partial(X_1 \cap X_2)| = |D(X_1 \cap X_2)|$ and $|\partial(X_1 \cup X_2| = |D(X_1 \cup X_2)|$.

2.2. Lemma. *Suppose that for each $X \subseteq V$, $|\partial(X)| \geq |D(X)|$. For critical sets $X_1, X_2 \subseteq V$ and for the set Y of vertices of a component of $G-(X_1 \cup X_2)$, if $D(X_1 \cap X_2;\, Y) = \emptyset$, then*

$$D(Y;\, V-(X_1 \cup X_2 \cup Y)) = \emptyset.$$

Proof. Put $Y_1 = V-(X_1 \cup X_2 \cup Y)$, then by simple counting we have

$$\begin{aligned}&|\partial(Y_1 \cup (X_1 - X_2))| + |\partial(Y_1 \cup (X_2 - X_1))| \\ &\quad = |\partial(X_1)| + |\partial(X_2)| + 2|\partial(Y;\, Y_1)| - 2|\partial(X_1 \cap X_2;\, Y)|,\end{aligned}$$

and a similar equality holds with D instead of ∂. Thus

$$\begin{aligned}&|\partial(Y_1 \cup (X_1 - X_2))| + |\partial(Y_1 \cup (X_2 - X_1))| \\ &\quad = |\partial(X_1)| + |\partial(X_2)| - 2|\partial(X_1 \cap X_2;\, Y)| \\ &\quad \geq |D(Y_1 \cup (X_1 - X_2))| + |D(Y_1 \cup (X_2 - X_1))| \\ &\quad = |D(X_1)| + |D(X_2)| + 2|D(Y;\, Y_1)|.\end{aligned}$$

Hence $D(Y;\, Y_1) = \emptyset$.

From [3] we have

2.3. Lemma. *$|\partial(X)| - |D(X)|$ is even for all $X \subseteq V$ if and only if it is even for all $X \subseteq V$ with $|X| = 1$.*

2.4. Lemma. *If G is connected, $|\partial(X)| \geq |D(X)|$ for all $X \subseteq V$ if and only if it holds for all elemental $X \subseteq V$.*

To prove Theorems 1.6 and 1.7 we use induction on $|E|$. The results are certainly true when $|E| = 0$ or $k = 0$, and so we assume that $|E| > 0$ and $k > 0$. We may assume that G is connected, and indeed 2-connected. Thus (unless $|E| = 1$, when the results are obvious) the boundary of the infinite face of G consists of a circuit C, which we regard as a subgraph of G. Clearly we may assume that $s_i \neq t_i$ for each i.

3. Proof of Theorem 1.6

If $l = 0$ or k, then Theorem 1.6 holds by [3], and so we may assume that $0 < l < k$. Let C_1 be the boundary of the face on which $s_1, \ldots, s_l, t_1, \ldots, t_l$ can be drawn. Then C_1 is a circuit. If $X \subseteq V$ is elemental, then clearly $|\partial(X) \cap E(C)| = 0$ or 2 and $|\partial(X) \cap E(C_1)| = 0$ or 2. We require the following lemmas.

3.1. Lemma. *Suppose that for a critical set $X \subseteq V$, $\partial(X)$ has just two edges of C and no edge of C_1. Then X is elemental.*

Proof. Assume that one of $\langle X\rangle, \langle V-X\rangle$ is not connected, say $\langle X\rangle$; and put $X=X_1\cup X_2$, where $X_1\cap X_2=\emptyset\neq X_i$ $(i=1,2)$ and $\partial(X_1;X_2)=\emptyset$. But $X\cap V(C)\subseteq X_i$ ($i=1$ or 2), say $i=1$; and so $D(X)=D(X_1)$. Now

$$|\partial(X_2)|=|\partial(X_2;X_1)|+|\partial(X_2;V-X)|=|\partial(X_2;V-X)|\neq\emptyset,$$

and

$$|\partial(X_1)|=|\partial(X_1;V-X)|=|\partial(X)|-|\partial(X_2;V-X)|.$$

Thus $|\partial(X_1)|<|\partial(X)|=|D(X)|=|D(X_1)|$, a contradiction.

3.2. Lemma. *For every $e\in E(C)$ with ends a,b (say) and every j $(l+1\leq j\leq k)$ we may assume that there is a critical elemental $X\subsetneq V$ with $s_j,t_j\notin X$ and with $X\cap\{a,b\}\neq\emptyset$.*

Proof. We take $j=k$ for definiteness. Let G' be $(V,E-\{e\})$, and let $k'=k+1$, and define $s_1',\dots,s_{k'}',t_1',\dots,t_{k'}'$ by

$$\begin{aligned}
&s_i'=s_i, \qquad && t_i'=t_i \qquad && (1\leq i\leq k-1),\\
&s_k'=s_k, && t_k'=b, &&\\
&s_{k+1}'=a, && t_{k+1}'=t_k. &&
\end{aligned}$$

Let D',∂' be the corresponding functions. Now by Lemma 2.3

$$|\partial'(X)|-|D'(X)|$$

is even for all $X\subseteq V$. If it is non-negative for all $X\subseteq V$, then by induction on $|E|$, G' has k' edge-disjoint paths P_i' between s_i' and t_i' $(1\leq i\leq k')$. We may combine P_k', e, and P_{k+1}', to make a path of G between s_k and t_k which is still edge-disjoint from $P_1',\dots,P_{k-1}'$; and 1.6 would be proved. Thus we may assume (for a contradiction) that for some $X\subsetneq V$,

$$|\partial'(X)|-|D'(X)|<0.$$

But this number is even, and clearly

$$|\partial'(X)|-|D'(X)|\geq|\partial(X)|-|D(X)|-3,$$

because only one edge of G has been deleted and only two extra pairs (s_i,t_i) have been added. Thus

$$|\partial'(X)|-|D'(X)|=-2 \quad\text{and}\quad |\partial(X)|-|D(X)|=0.$$

Moreover by Lemma 2.4 we may assume that X is elemental in G', and therefore that X is elemental in G. Further, by replacing X by $V-X$ if necessary, we may assume that $s_k\notin X$. There therefore remain four possibilities:

(i) $a,b\in X, s_k,t_k\notin X$.
(ii) $a\in X, b,s_k,t_k\notin X$.
(iii) $b\in X, a,s_k,t_k\notin X$.

(iv) $b, t_k \in X, a, s_k \notin X$.

If any of (i), (ii), (iii) holds then the lemma is proved; thus we suppose, for a contradiction, that they are all false and that (iv) holds. By the same argument applied with a, b exchanged, we deduce that there is a critical elemental set $X' \subseteq V$ with $a, t_k \in X'$, $b, s_k \notin X'$. Thus in particular a, b, s_k, t_k are all distinct. X is elemental, and so $\langle V(C) \cap X \rangle$ and $\langle V(C) - X \rangle$ are paths; thus there are two disjoint paths of C joining b to t_k and a to s_k. Similarly, since X' is elemental, there are two disjoint paths of C joining a to t_k and b to s_k. But this is impossible, since C is a circuit and a, b are adjacent in C. That completes the proof of Lemma 3.2.

Let e_1, e_2 be the edges incident to s_k in C and let a_i be another end of e_i $(i = 1, 2)$. By Lemma 3.2 we may assume that there is a critical elemental $X_i \subseteq V$ with $s_k, t_k \notin X_i$ and with $a_i \in X_i$ $(i = 1, 2)$. Choose X_i with this property, such that $V(C) \cap X_i$ is minimal $(i = 1, 2)$. If for $i = 1, 2$ there exists a vertex v_i in $V(C_1) \cap X_i$, then there exists a path H_i between a_i and v_i in $\langle X_i \rangle$, since X_i is elemental. Since s_k, t_k are different vertices in $C - (X_1 \cup X_2)$, every path between s_k and t_k intersects H_1 or H_2. Therefore s_k and t_k are contained in different components of $G - (X_1 \cup X_2)$. Let Y_1, Y_2 be the sets of vertices of the components of $G - (X_1 \cup X_2)$ which contains s_k, t_k respectively.

We will show that either $D(X_1 \cap X_2;\, Y_1) = \emptyset$ or $D(X_1 \cap X_2;\, Y_2) = \emptyset$. For $i = 1, 2$ $\langle V(C) \cap X_i \rangle$ is a path, since X_i is elemental. If $V(C) \cap X_1 \cap X_2 \neq \emptyset$, then $\langle V(C) \cap (X_1 \cup X_2) \rangle$ and $\langle V(C) - (X_1 \cup X_2) \rangle$ are paths, contradicting that s_k, t_k are contained in different components of $C - (X_1 \cup X_2)$. Thus $V(C) \cap X_1 \cap X_2 = \emptyset$. If $V(C_1) \cap X_1 \cap X_2 = \emptyset$, then the result follows, and so we assume that $V(C_1) \cap X_1 \cap X_2 \neq \emptyset$. For $i = 1, 2$ $\langle V(C_1) \cap X_i \rangle$ is a path or C_1, since X_i is elemental and $V(C_1) \cap X_i \neq \emptyset$. If $V(C_1) \subseteq X_1 \cup X_2$, then $V(C_1) \cap Y_1 = V(C_1) \cap Y_2 = \emptyset$. If $V(C_1) \not\subseteq X_1 \cup X_2$, then $\langle V(C_1) - (X_1 \cup X_2) \rangle$ is a path, and so $V(C_1) \cap Y_1 = \emptyset$ or $V(C_1) \cap Y_2 = \emptyset$. Therefore either $D(X_1 \cap X_2;\, Y_1) = \emptyset$ or $D(X_1 \cap X_2;\, Y_2) = \emptyset$. Now $D(Y_1;\, Y_2) = \emptyset$ by Lemma 2.2, contradicting that $k \in D(Y_1;\, Y_2)$. Hence $V(C_1) \cap X_i = \emptyset$ for $i = 1$ or $i = 2$, say $i = 1$. Then there exists $c \in V(C) - X_1$ such that for some $d \in V(C) \cap X_1$ and some i $(l + 1 \leq i \leq k)$ $(s_i, t_i) = (c, d)$ or (d, c). Choose c with this property, such that the subpath of C from s_k to c not using a_1 has minimum length. By Lemma 3.2 there is a critical elemental $X_3 \subseteq V$ with $c, d \notin X_3$ and with $\{a_1, s_k\} \cap X_3 \neq \emptyset$. Then $V(C) \cap (X_3 - X_1)$ is included in the set of vertices of the subpath of C between s_k and c not using a_1, and does not contain c; and so by choice of c and since $V(C_1) \cap X_1 = \emptyset$, we have $D(X_1 - X_3;\, X_3 - X_1) = \emptyset$. Hence by Lemma 2.1 $\partial(X_1 - X_3;\, X_3 - X_1) = \emptyset$ and in particular does not contain e_1; and so $X_3 \cap \{a_1, s_k\} \neq \{s_k\}$. Hence $a_1 \in X_3$. Moreover by Lemma 2.1 $X_1 \cap X_3$ is critical, and $\partial(X_1 \cap X_3)$ contains just two edges of C and no edge of C_1. Thus by Lemma 3.1 $X_1 \cap X_3$ is elemental, and

$$a_1 \in X_1 \cap X_3 \subseteq X_1 - \{d\}.$$

This contradicts the minimality of $X_1 \cap V(C)$, as required.

4. Proof of Theorem 1.7

Let $e_1, \dots, e_n$ be the edges incident to s_1 and a_i be the other end of e_i $(1 \le i \le n)$. We require the following lemma.

4.1. Lemma. *We may assume that for every i $(1 \le i \le n)$ there exists a critical elemental $Z_i \subseteq V$ with $s_1, t_1 \notin Z_i$ and with $a_i \in Z_i$.*

We can prove Lemma 4.1 in the same way as Lemma 3.2. (e_i or s_1 might not be in C, but note that s_1, t_1 and the two ends of e_i are not all distinct.) We use Lemma 4.1 to obtain a contradiction.

We will show that there are critical sets $X_1, X_2 \subseteq V$ such that $X_1 \cup X_2$ separates s_1 and t_1 and such that $D(X_1 \cap X_2;\ Y) = \emptyset$, where Y is the set of vertices of the component of $G-(X_1 \cup X_2)$ which contains s_1. Then by Lemma 2.2 it follows that $D(Y;\ V-(X_1 \cup X_2 \cup Y)) = \emptyset$, contradicting that this contains 1, as required.

If every $\partial(Z_i)$ or except one does not contain an edge of C, let $\partial(Z_i) \cap E(C) = \emptyset$ $(2 \le i \le n)$. Then clearly $D(Z_2; Z_3) = \emptyset$ and, by Lemma 2.1, $Z_2 \cup Z_3$ is critical, and by repeating this it follows that $\bigcup_{2 \le i \le n} Z_i$ is critical. Then we can choose Z_1, $\bigcup_{2 \le i \le n} Z_i$ as X_1, X_2 respectively, since $Y = \{s_1\} \not\ni t_1$ and $D(X_1 \cap X_2;\ Y) = \emptyset$. Thus without loss of generality we may assume that for some $m \ge 2, \partial(Z_1), \dots, \partial(Z_m)$ contain just two edges of C and $\partial(Z_{m+1}), \dots, \partial(Z_n)$ contain no edge of C.

We require the following lemma.

4.2. Lemma. *For some p, q $(1 \le p, q \le m)$ $Z_p \cup Z_q \cup Z_{m+1} \cup \cdots \cup Z_n$ separates s_1 and t_1.*

Proof. For $1 \le i \le m$ there exists a path H_i between a_i and a vertex of $V(C) \cap Z_i$ in $\langle Z_i \rangle$, since Z_i is elemental; choose H_i with this property such that H_i has minimum length. Then H_i is a simple path and $|V(C) \cap V(H_i)| = 1$, and let b_i be in $V(C) \cap V(H_i)$ $(1 \le i \le m)$. $C \cup H_1 \cup H_2 \cup \cdots \cup H_m \cup \{e_1, \dots, e_m\}$ divides the interior of C into regions. Let R be such a region which contains t_1 on its boundary $\bar{R}$. If s_1 is on $\bar{R}$, then for just two p, q $(1 \le p < q \le m)$ e_p, e_q are on $\bar{R}$. Now $Z_p \cup Z_q \cup Z_{m+1} \cup \cdots \cup Z_n$ separates s_1 and t_1, since the component of $G-(Z_p \cup Z_q \cup \{s_1\})$ which contains t_1 does not contain any a_i $(1 \le i \le m)$. We assume that s_1 is not on $\bar{R}$, then $e_i \notin \bar{R}$ $(1 \le i \le m)$. Let K_i $(1 \le i \le r)$ be a subpath of H_{α_i} for some α_i $(1 \le \alpha_i \le m)$ and let K_0 be a subpath of C such that $K_0 \cup K_1 \cup \cdots \cup K_r = \bar{R}$. We assume that $K_0, K_1, \dots, K_r$ are in this order on $\bar{R}$, for $1 \le i \le r-1$ $\alpha_i \ne \alpha_{i+1}$ and that for i, j $(0 \le i \le r-1, j = i+1;\ i = r, j = 0)$ $V(K_i) \cap V(K_j) = \{v_i\}$. If $r = 2$, then we can choose α_1, α_2 as p, q; and so we may assume that $r \ge 3$. For $2 \le i \le r-1$ let A_i, B_i be subpaths of H_{α_i} such that

$$A_i \cup K_i \cup B_i = H_{\alpha_i}, \qquad V(A_i) \cap V(K_i) = \{v_{i-1}\},$$
$$V(K_i) \cap V(B_i) = \{v_i\} \quad \text{and} \quad V(A_i) \cap V(B_i) = \emptyset.$$

If $b_{\alpha_2} \in V(B_2)$, then the component of $G-(K_1 \cup K_2 \cup B_2)$ which contains t_1 does not contain any a_i $(1 \leq i \leq m)$, since H_{α_2} is a simple path and $s_1 \notin V(H_{\alpha_2})$. Therefore we can choose α_1, α_2 as p, q. If for some j $(2 \leq j \leq r-2)$ $b_{\alpha_i} \notin V(B_i)$ $(2 \leq i \leq j)$ and $b_{\alpha_{j+1}} \in V(B_{j+1})$, then we can choose α_j, α_{j+1} as p, q. If $b_{\alpha_i} \notin V(B_i)$ $(2 \leq i \leq r-1)$, then we can choose α_{r-1}, α_r as p, q.

Now we will show that we can choose $Z_p, Z_q \cup Z_{j+1} \cup \cdots \cup Z_n$ as X_1, X_2. For as before $\bigcup_{j+1 \leq i \leq n} Z_i$ is critical. Since $Z_q \not\ni t_1 = \cdots = t_l$, $D(\bigcup_{j+1 \leq i \leq n} Z_i; Z_q) = \emptyset$ and by Lemma 2.1 $Z_q \cup Z_{j+1} \cup \cdots \cup Z_n$ is critical. Now either $X_1 \cap X_2 \cap V(C) = \emptyset$ or $Y \cap V(C) = \emptyset$. For if not, then $C-(X_1 \cup X_2) = C-(Z_p \cup Z_q)$ is a path and there exists a vertex v in $Y \cap V(C)$, and so $v, t_1 \in C-(X_1 \cup X_2)$ and v, t_1 contained in one component of $G-(X_1 \cup X_2)$, contradicting that $t_1 \notin Y$. Moreover t_1 is contained in $V-(X_1 \cup X_2 \cup Y)$, and so $D(X_1 \cap X_2; Y) = \emptyset$. This completes the proof.

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